Graph Colouring Is Hard on Average for Polynomial Calculus

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Proof Complexity and Beyond
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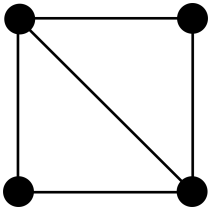
Joint work with Susanna F. de Rezende, Jakob Nordström, Shuo Pang, and Kilian Risse

Graph Colouring

Can vertices of graph G be coloured with k colours so that no edge is monochromatic?

One of Karp's original 21 NP-complete problems [Kar72]



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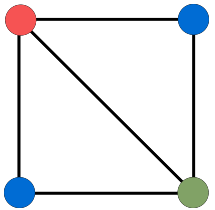

✓: $k = 3$

✗: $k = 2$

Is Colouring Hard?

On one hand, colouring is hard—even to approximate:

- if G is k-colourable, best efficient algorithm uses $kn/\mathrm{polylog}(n)$ colours [Hal93]
- if G promised 3-colourable, best efficient algorithm uses $n^{0.199..}$ colours [KT17]
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...but practical algorithms often perform surprisingly well, e.g.

- backtracking search [Kor75; Lew21]
- integer programming [MT96; GM12]
- algebraic algorithms [DLMM08; DLMO09; DLMM11; DMP+15]

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Algebraic algorithms captured by algebraic proof systems

Proof complexity lower bounds $\implies$ unconditional hardness for these algorithms

Our Results

For algebraic proof systems, *worst-case* exponential lower bounds known for colouring [LN17; AO19]

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Colouring *easy* except in few artificial cases?

To refute this, want average-case hardness, just as for resolution [BCMM05]

Main Result

With probability $1 - o(1)$, *polynomial calculus* requires exponential size for refuting 3-colouring on random graphs

Polynomial Calculus [CEI96]

To prove set of polynomials $\mathcal{P} = \{p_1, \ldots, p_m\}$ has no common root, derive new polynomials in ideal $\langle \mathcal{P} \rangle$ through

$$\text{Linear combination: } \frac{p \quad q}{\alpha p + \beta q} \qquad \alpha, \beta \in \mathbb{F}$$

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Complexity measures:

- **Size:** Total # of monomials in proof lines (with multiplicities)
- **Degree:** Largest degree among monomials in proof lines

Graph Colouring and Polynomials

Encode k-colouring as polynomials over field $\mathbb{F}$

$x_{v,i} = 1 \iff$ "vertex v gets colour i"

$\sum_{i=1}^{k} x_{v,i} - 1,$	$\forall v$	"every vertex gets a colour"
$x_{v,i} \cdot x_{v,i'},$	$\forall v,\ i \neq i'$	"no vertex gets > 1 colour"
$x_{u,i} \cdot x_{v,i},$	$\forall (u,v) \in E(G)$	"no monochromatic edges"
$x_{v,i}^2 - x_{v,i},$	$\forall v, i$	Boolean axioms

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Can also deal with other encoding [Bay82] more common in math:

- Add kth root of unity ξ to $\mathbb{F}$
- $x_v = \xi^i \iff$ “vertex v gets colour i”

Formal Statement of Main Result

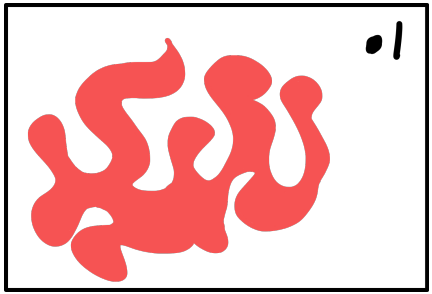
Theorem

If G is sparse random graph on n vertices, then with probability $1 - o(1)$ polynomial calculus requires size $\exp(\Omega(n))$ to refute G is 3-colourable.

- Holds over any field
- Holds for both random regular graphs and Erdős–Rényi random graphs

Prove $\Omega(n)$ degree lower bound; implies $\exp(\Omega(n))$ size lower bound [IPS99]

Degree Lower Bounds and R-operators



Derivable in degree $\leq D$

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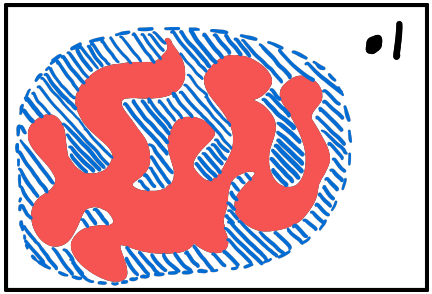

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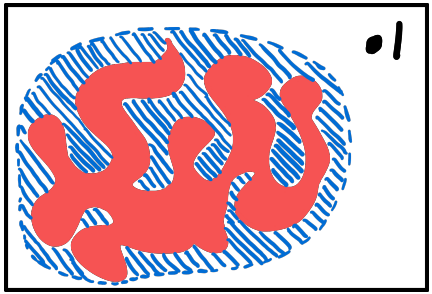

Derivable in degree $\leq D$

Overapproximation

Define so-called R**-operator** [Raz98] on polynomials such that

- $R(p) = 0$, for each input polynomial p
- $R(p) + R(q) = R(p + q)$
- If $R(p) = 0$ then $R(x \cdot p) = 0$, for all p of degree $\leq D - 1$
- $R(1) = 1$

Overapproximation is kernel of R

R as in Reduction

Put total order $\prec$ on monomials in $\mathbb{F}[\mathbf{x}]$, where 1 smallest

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Ideal $\langle \mathcal{P} \rangle$ of $\mathcal{P} = \{p_1, \ldots, p_m\}$ is set of polynomials $q = \sum_i q_i p_i$

For ideal $\langle \mathcal{P} \rangle$, define **reduction operator** $R_{\langle \mathcal{P} \rangle} : p \mapsto r$

- r is polynomial with smallest terms such that $r = p - q$, where $q \in \langle \mathcal{P} \rangle$
- analogous to remainder term after division

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For unsatisfiable input, *pseudo-reduction* operator R pretends to be above reduction operator. Low-degree computations cannot tell the difference.

Proof Ideas

If set $\mathcal{P}$ of input polynomials satisfiable, get *perfect* R-operator from reduction modulo $\langle \mathcal{P} \rangle$

...but $\mathcal{P}$ unsatisfiable, so $1 \in \langle \mathcal{P} \rangle$

“Almost” works. Still leverage reduction somehow?

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Alekhnovich–Razborov [AR03]

- define R using real reduction
- reduce different monomials modulo different satisfiable subsets of $\mathcal{P}$
- carefully choose subsets so inconsistencies invisible in low degree

Local” ReductionLocal” ReductionLocal” ReductionLocal” ReductionLocal” ReductionLocal” Reduction“Local” Reduction

In more detail, idea is:

1. Associate $m \sim S(m) \subseteq V$ and ideal $\langle S(m) \rangle$ generated by k-colouring polynomials on $G[S(m)]$
2. Define R “locally” on each monomial:

$$R(p) = R\Big(\sum_i a_i m_i\Big) \coloneqq \sum_i a_i R_{\langle S(m_i) \rangle}(m_i)$$

Our Goal

Want R to look like reduction modulo ideal for low-degree p

Maybe, for well-chosen S and, say, $p = m_1 + m_2$, could get

$$\begin{aligned} R(m_1 + m_2) &= R_{\langle S(m_1)\rangle}(m_1) + R_{\langle S(m_2)\rangle}(m_2) \\ &\stackrel{!}{=} R_{\langle S(m_1)\cup S(m_2)\rangle}(m_1 + m_2) \end{aligned}$$

That is, want $R_{\langle S(m)\rangle}(m) = R_{\langle U\rangle}(m)$ for all $U \supseteq S(m)$ not too large

Our Goal

What does it mean that $R_{\langle S(m)\rangle}(m) = R_{\langle U\rangle}(m)$?

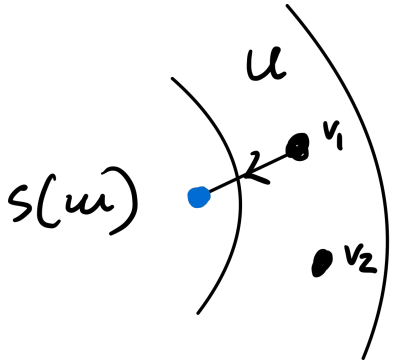
Syntactically: best reduction of m by $\langle U\rangle$ could be done already in $\langle S(m)\rangle$

Our Goal

Semantically: put order on V, can extend every colouring of $S(m)$ to one for U in order-preserving way

$\Longrightarrow$ "U says no more than $S(m)$" about colourings of m

Order-preserving: colours in $U \setminus S(m)$ either fixed or depend only on single, smaller vertex in $S(m)$



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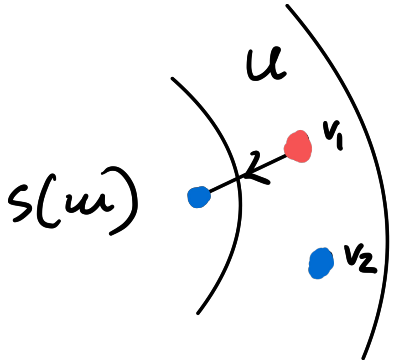

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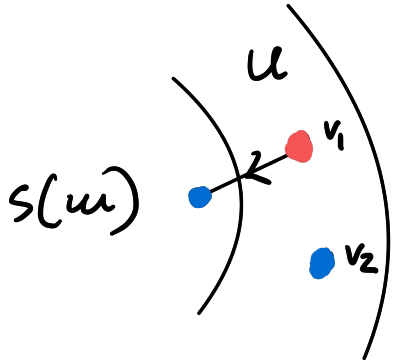
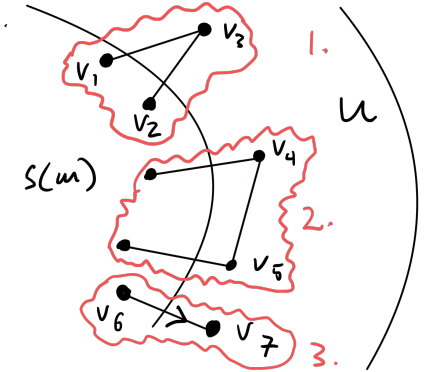
Obstructions?



Constructing $S(m)$

Three obstructions:

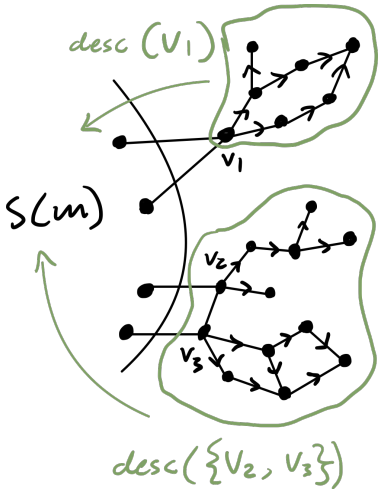
1. dependence on > 1 vertex in $S(m)$
2. dependence between neighbours of $S(m)$
3. small neighbours



Constructing $S(m)$

Construct $S(m)$ iteratively:

1. start with $S(m) = \mathrm{Desc}(V(m))$
2. while bad structure exists, add it and descendants to $S(m)$



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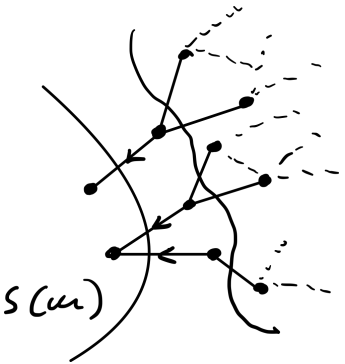

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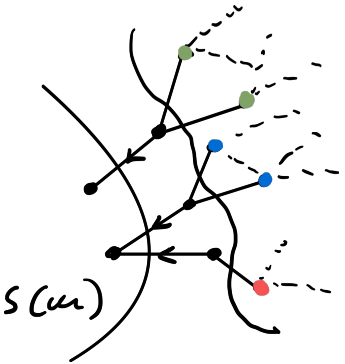

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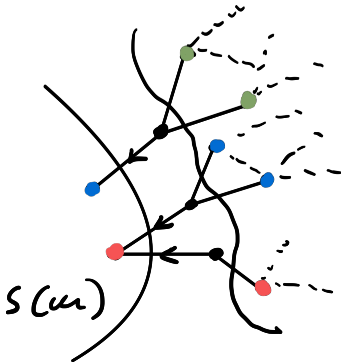

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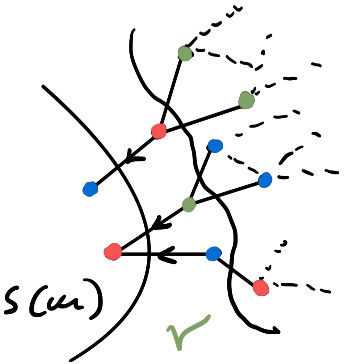
But not clear at all size of $S(m)$ does not blow up...



Key Technical Ingredients

1 **Local sparsity:** Vertex-induced subgraph of every subset $U \subseteq V$ of size $\leq \varepsilon n$ has at most $(1+\delta)|U|$ edges

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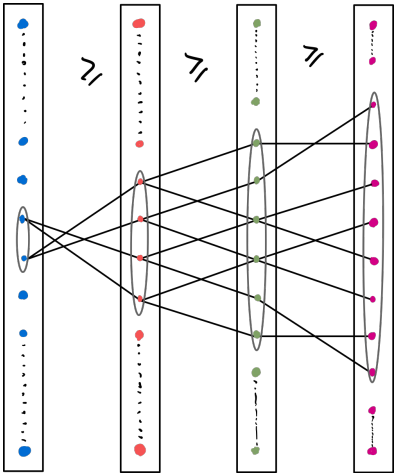
2 **Good vertex order:**

- always add all descendants, so this set must be small for every vertex
- if all ordered paths have length c and max degree is Δ, size is at most Δ^c

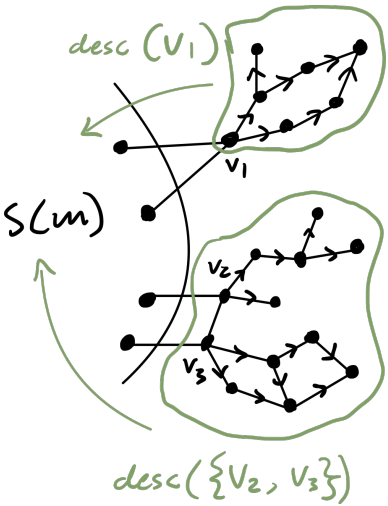
[RT22]: order by proper colouring of graph $\implies$ ordered paths have length $\chi(G) = O(1)$



$S(m)$ Is Small

Proof by picture:

- at each step, add $\geq$ one more edge than vertices
- short ordered paths $\Longrightarrow$ few vertices added per step
- quickly becomes dense—contradicts sparsity



Open Problems

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   - Sherali–Adams
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2. Results field-independent; refine to account for characteristic (cf. [AR03])?

Summary

This work:

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